Intermediate Value Theorem

8. State the Intermediate Value Theorem.

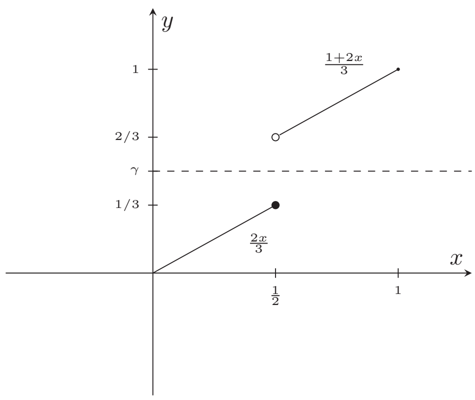
   Give an example of a strictly increasing function f on $[0,1]$ and a value $\gamma : f(0) < \gamma < f(1)$ for which there is **not** a $c \in [0,1]$ with $f(c) = \gamma$.

   Solution ***Intermediate Value Theorem*** *Suppose that f is a function continuous on a closed interval $[a,b]$ and that $f(a) \neq f(b)$. For all γ between $f(a)$ and $f(b)$ there exist $c : a < c < b$ for which $f(c) = \gamma$.*

   For the required example: choose $f : [0,1] \to [0,1]$,

$$x \longmapsto \begin{cases} \dfrac{2x}{3} & 0 \le x \le \frac{1}{2} \\ \dfrac{1+2x}{3} & \frac{1}{2} < x \le 1. \end{cases}$$

   The image of this function is $[0,1/3] \cup (2/3,1]$. Choose $\gamma = 1/2$. There is no c for which $f(c) = 1/2$.



   The point of this question is, does the conclusion of the Intermediate Value Theorem follow if we weaken the assumption, i.e. if we do not assume f is continuous. Answer NO. In this example we have a *non-continuous* function for which the conclusion does not follow.

9. Show that

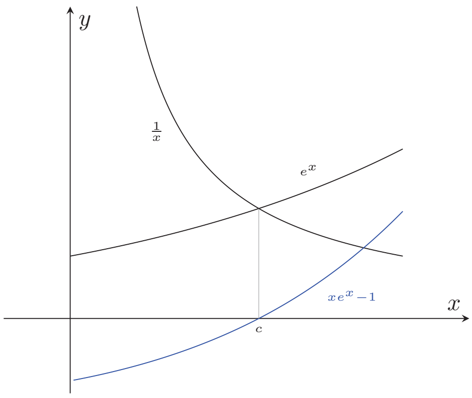
$$e^x = \frac{1}{x}$$

   has **a** solution in $[0,1]$.

Solution Multiply up and let $f(x) = xe^x - 1$. Then $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$. Thus 0 is intermediate between $f(0)$ and $f(1)$, i.e. $f(0) < 0 < f(1)$. So by the Intermediate Theorem with $\gamma = 0$ there exists $c \in (0,1)$ such that $f(c) = \gamma = 0$, i.e. $ce^c - 1 = 0$. Since $c > 0$ we can rearrange and divide by c to get

$$e^c = \frac{1}{c}.$$

Plots of $y = 1/x$, $y = e^x$ and $y = xe^x - 1$:



Note if you were to define $f(x) = e^x - 1/x$, then $f(0)$ would not be defined. Instead we follow the principle of ridding ourselves of fractions whenever possible. **End of Note**

10. Show that $e^x = 4x^2$ has **at least three** real solutions.

Solution Let $f(x) = e^x - 4x^2$. You have to look at random intervals trying to find sign changes. As example, at the points $x = -1, 0, 1$ and 8 we find

$$\begin{aligned} f(-1) &= e^{-1} - 4 \times (-1)^2 < 0 \\ f(0) &= e^0 - 4 \times 0^2 = 1 > 0, \\ f(1) &= e^1 - 4 < 0, \\ f(8) &= e^8 - 4 \times 8^2 > 2^8 - 2^8 = 0. \end{aligned}$$

(In the last line I have used the weak (but strict) lower bound of $e > 2$). Thus there are sign changes in each of the intervals $[-1, 0]$, $[0, 1]$ and $[1, 8]$ and hence a zero in each interval.

Plot of $y = e^x - 4x^2$